

HOPF STRUCTURES ON THE MULTIPLIHEDRA

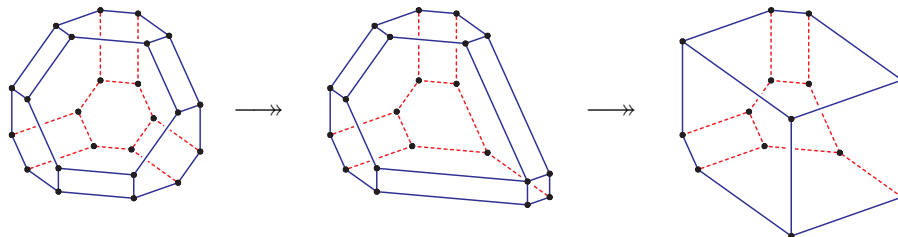
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ABSTRACT. We investigate algebraic structures that can be placed on vertices of the multiplihedra, a family of polytopes originating in the study of higher categories and homotopy theory. Most compelling among these are two distinct structures of a Hopf module over the Loday–Ronco Hopf algebra.

INTRODUCTION

The permutahedra \mathfrak{S}_n form a family of highly symmetric polytopes that have been of interest since their introduction by Schoute in 1911 [23]. The associahedra \mathcal{Y}_n are another family of polytopes that were introduced by Stasheff as cell complexes in 1963 [25], and with the permutahedra were studied from the perspective of monoidal categories and H -spaces [17] in the 1960s. Only later were associahedra shown to be polytopes [11, 13, 18]. Interest in these objects was heightened in the 1990s, when Hopf algebra structures were placed on them in work of Malvenuto, Reutenauer, Loday, Ronco, Chapoton, and others [6, 14, 16]. More recently, the associahedra were shown to arise in Lie theory through work of Fomin and Zelevinsky on cluster algebras [7].

We investigate Hopf structures on another family of polyhedra, the multiplihedra, \mathcal{M}_n . Stasheff introduced them in the context of maps preserving higher homotopy associativity [26] and described their 1-skeleta. Boardman and Vogt [5], and then Iwase and Mimura [12] described the multiplihedra as cell complexes, and only recently were they shown to be convex polytopes [8]. These three families of polytopes are closely related. For each integer $n \geq 1$, the permutahedron \mathfrak{S}_n , multiplihedron \mathcal{M}_n , and associahedron \mathcal{Y}_n are polytopes of dimension $n-1$ with natural cellular surjections $\mathfrak{S}_n \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \mathcal{Y}_n$, which we illustrate when $n = 4$.



The faces of these polytopes are represented by different flavors of planar trees; permutahedra by ordered trees (set compositions), multiplihedra by bi-leveled trees (Section 2.1), and associahedra by planar trees. The maps between them forget

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the additional structure on the trees. These maps induce surjective maps of graded vector spaces spanned by the vertices, which are binary trees. The span $\mathfrak{S}Sym$ of ordered trees forms the Malvenuto-Reutenauer Hopf algebra [16] and the span $\mathcal{Y}Sym$ of planar binary trees forms the Loday-Ronco Hopf algebra [14]. The algebraic structures of multiplication and comultiplication on $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ are described in terms of geometric operations on trees and the composed surjection $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$ gives a surjective morphism $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$ of Hopf algebras.

We define $\mathcal{M}Sym$ to be the vector space spanned by the vertices of all multiplihedra. The factorization of τ induced by the maps of polytopes, $\mathfrak{S}Sym \rightarrow \mathcal{M}Sym \rightarrow \mathcal{Y}Sym$, does not endow $\mathcal{M}Sym$ with the structure of a Hopf algebra. Nevertheless, some algebraic structure does survive the factorization. We show in Section 3 that $\mathcal{M}Sym$ is an algebra, which is simultaneously a $\mathfrak{S}Sym$ -module and a $\mathcal{Y}Sym$ -Hopf module algebra, and the maps preserve these structures.

We perform a change of basis in $\mathcal{M}Sym$ using Möbius inversion that illuminates its comodule structure. Such changes of basis helped to understand the coalgebra structure of $\mathfrak{S}Sym$ [1] and of $\mathcal{Y}Sym$ [2]. Section 4 discusses a second $\mathcal{Y}Sym$ Hopf module structure that may be placed on the positive part $\mathcal{M}Sym_+$ of $\mathcal{M}Sym$. This structure also arises from polytope maps between \mathfrak{S}_\bullet and \mathcal{Y}_\bullet , but not directly from the algebra structure of $\mathfrak{S}Sym$. Möbius inversion again reveals an explicit basis of $\mathcal{Y}Sym$ coinvariants in this alternate setting.

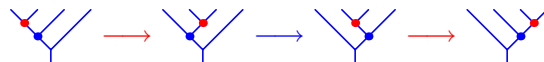
1. BASIC COMBINATORIAL DATA

The structures of the Malvenuto-Reutenauer and Loday-Ronco algebras are related to the weak order on ordered trees and the Tamari order on planar trees. There are natural maps between the weak and Tamari orders which induce a morphism of Hopf algebras. We first recall these partial orders and then the basic structure of these Hopf algebras. In Section 1.3 we establish a formula involving the Möbius functions of two posets related by an interval retract. This is a strictly weaker notion than that of a Galois correspondence, which was used to study the structure of the Loday-Ronco Hopf algebra.

1.1. \mathfrak{S}_\bullet and \mathcal{Y}_\bullet . The 1-skeleta of the families of polytopes \mathfrak{S}_\bullet , \mathcal{M}_\bullet , and \mathcal{Y}_\bullet are Hasse diagrams of posets whose structures are intertwined with the algebra structures we study. We use the same notation for a polytope and its poset of vertices. Similarly, we use the same notation for a cellular surjection of polytopes and the poset map formed by restricting that surjection to vertices.

For the permutahedron \mathfrak{S}_n , the corresponding poset is the (left) *weak order*, which we describe in terms of permutations. A cover in the weak order has the form $w \lessdot (k, k+1)w$, where k precedes $k+1$ among the values of w . Figure 1 displays the weak order on \mathfrak{S}_4 . We let $\mathfrak{S}_0 = \{\emptyset\}$, where \emptyset is the empty permutation of \emptyset .

Let \mathcal{Y}_n be the set of rooted, planar binary trees with n nodes. The cover relations in the *Tamari order* on \mathcal{Y}_n are obtained by moving a child node directly above a given node from the left to the right branch above the given node. Thus



is an increasing chain in \mathcal{Y}_3 (the moving vertices are marked with dots). Figure 1 shows the Tamari order on \mathcal{Y}_4 .

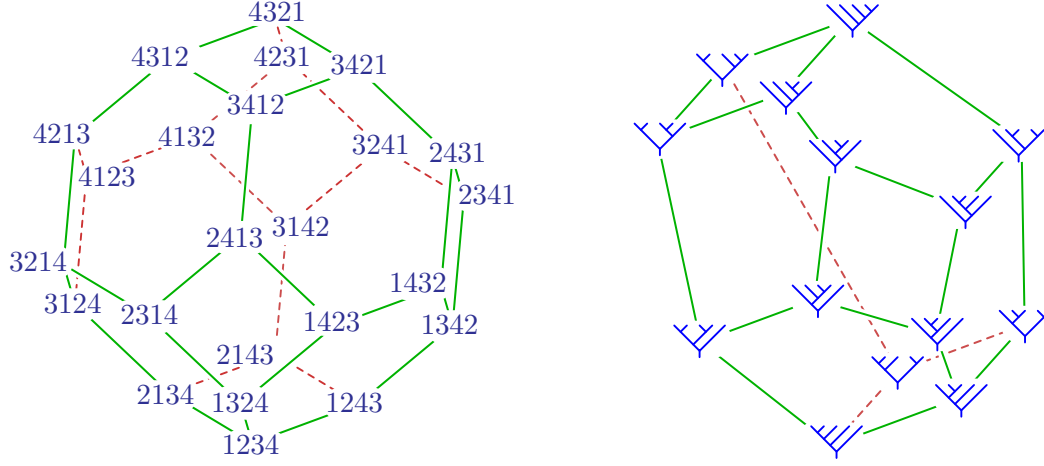
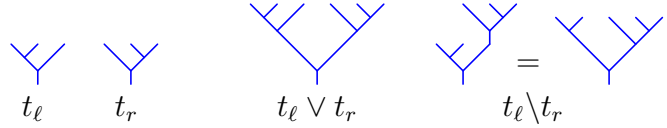


FIGURE 1. Weak order on \mathfrak{S}_4 and Tamari order on \mathcal{Y}_4

The unique tree in \mathcal{Y}_1 is Υ . Given trees t_ℓ and t_r , form the tree $t_\ell \vee t_r$ by grafting the root of t_ℓ (respectively of t_r) to the left (respectively right) leaf of Υ . Form the tree $t_\ell \setminus t_r$ by grafting the root of t_r to the rightmost leaf of t_ℓ . For example,



Decompositions $t = t_1 \setminus t_2$ correspond to pruning t along the right branches from the root. A tree t is *indecomposable* if it has no nontrivial decomposition $t = t_1 \setminus t_2$ with $t_1, t_2 \neq \mathbf{1}$. Equivalently, if the root node is the rightmost node of t . Any tree t is uniquely decomposed $t = t_1 \setminus \dots \setminus t_m$ into indecomposable trees t_1, \dots, t_m .

We define a poset map $\tau : \mathfrak{S}_n \rightarrow \mathcal{Y}_n$. First, given distinct integers a_1, \dots, a_k , let $\bar{a} \in \mathfrak{S}_k$ be the unique permutation such that $\bar{a}(i) < \bar{a}(j)$ if and only if $a_i < a_j$. Thus $\overline{4726} = 2413$. Since $\mathfrak{S}_0, \mathcal{Y}_0, \mathfrak{S}_1$, and \mathcal{Y}_1 are singletons, we must have

$$\begin{aligned} \tau : \mathfrak{S}_0 &\longrightarrow \mathcal{Y}_0 \quad \text{with} \quad \tau : \emptyset \longmapsto \mathbf{1}, \quad \text{and} \\ \tau : \mathfrak{S}_1 &\longrightarrow \mathcal{Y}_1 \quad \text{with} \quad \tau : 1 \longmapsto \Upsilon. \end{aligned}$$

Let $n > 0$ and assume that τ has been defined on \mathfrak{S}_k for $k < n$. For $w \in \mathfrak{S}_n$ suppose that $w(j) = n$, and define

$$\tau(w) := \tau(\overline{w(1), \dots, w(j-1)}) \vee \tau(\overline{w(j+1), \dots, w(n)}).$$

For example,

$$\begin{aligned} \tau(12) &= \Upsilon \vee \mathbf{1} = \Upsilon, \quad \tau(21) = \mathbf{1} \vee \Upsilon = \Upsilon, \quad \text{and} \\ \tau(3421) &= \tau(\overline{3}) \vee \tau(\overline{21}) = \tau(1) \vee \tau(21) = \Upsilon \vee \Upsilon = \Upsilon. \end{aligned}$$

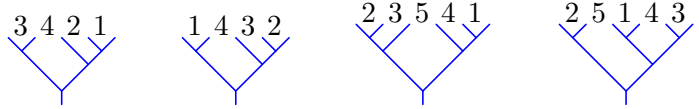
Loday and Ronco [15] show that the fibers $\tau^{-1}(t)$ of τ are intervals in the weak order. This gives two canonical sections of τ . For $t \in \mathcal{Y}_n$,

$$\mathbf{min}(t) := \min \{w \mid \tau(w) = t\} \quad \text{and} \quad \mathbf{max}(t) := \max \{w \mid \tau(w) = t\},$$

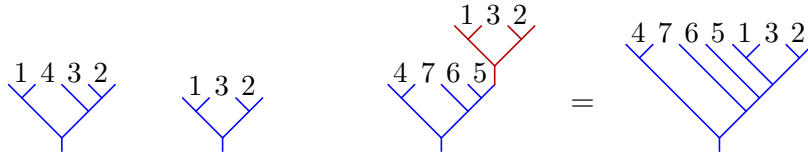
the minimum and maximum in the weak order. Equivalently, $\mathbf{min}(t)$ is the unique 231-avoiding permutation in $\tau^{-1}(t)$ and $\mathbf{max}(t)$ is the unique 132-avoiding permutation. These maps are order-preserving.

The 1-skeleta of \mathfrak{S}_n and \mathcal{Y}_n form the Hasse diagrams of the weak and Tamari orders, respectively. Since τ is an order-preserving surjection, it induces a cellular map between the 1-skeleta of these polytopes. Tonks [27] extended τ to the faces of \mathfrak{S}_n , giving a cellular surjection.

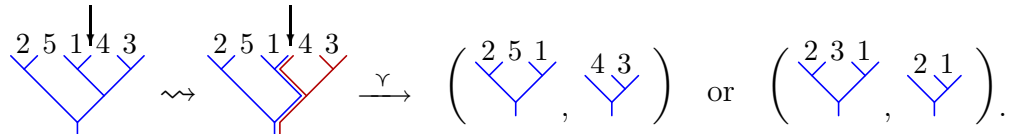
The nodes and internal edges of a tree are the Hasse diagram of a poset with the root node maximal. Labeling the nodes (equivalently, the gaps between the leaves) of $\tau(w)$ with the values of the permutation w gives a linear extension of the node poset of $\tau(w)$, and all linear extensions of a tree t arise in this way for a unique permutation in $\tau^{-1}(t)$. Such a linear extension w of a tree is an *ordered tree* and $\tau(w)$ is the corresponding unordered tree. In this way, \mathfrak{S}_n is identified with the set of ordered trees with n nodes. Here are some ordered trees,



Given ordered trees u, v , form the ordered tree $u \setminus v$ by grafting the root of v to the rightmost leaf of u , where the nodes of u are greater than the nodes of v , but the relative orders within u and v are maintained. Thus we may decompose an ordered tree $w = u \setminus v$ whenever $\tau(w) = r \setminus s$ with $\tau(u) = r$, $\tau(v) = s$, and the nodes of r in w precede the nodes of s in w . An ordered tree w is *indecomposable* if it has no nontrivial such decompositions. Here are ordered trees u, v and $u \setminus v$,



We may *split* an ordered tree w along a leaf to obtain either an ordered forest (where the nodes in the forest are totally ordered) or a pair of ordered trees,



Write $w \xrightarrow{\gamma} (w_0, w_1)$ to indicate that the ordered forest (w_0, w_1) (or pair of ordered trees) is obtained by splitting w along some leaf. (Context will determine how to interpret the result.) More generally, we may split an ordered tree w along a multiset of $m \geq 0$ of its leaves to obtain an ordered forest, or tuple of ordered

trees, written $w \xrightarrow{\gamma} (w_0, \dots, w_m)$. For example,

$$(1.1) \quad \begin{array}{c} \text{Diagram 1: A tree with root at bottom, left child has children 3, 2, right child has children 7, 5, 1, 6, 4.} \end{array} \rightsquigarrow \begin{array}{c} \text{Diagram 2: The same tree as Diagram 1, but with edges colored (blue, green, red, purple) to represent a splitting.} \end{array} \xrightarrow{\gamma} \left(\begin{array}{c} \text{Diagram 3: A tree with root at bottom, left child has children 3, 2, right child has children 7, 5, 1, 6, 4.} \end{array}, \begin{array}{c} \text{Diagram 4: A single edge.} \end{array}, \begin{array}{c} \text{Diagram 5: A tree with root at bottom, left child has children 7, 5, 1, right child has children 6, 4.} \end{array}, \begin{array}{c} \text{Diagram 6: A single edge.} \end{array}, \begin{array}{c} \text{Diagram 7: A single edge.} \end{array} \right).$$

Given $v \in \mathfrak{S}_m$ and an ordered forest (w_0, \dots, w_m) , let $(w_0, \dots, w_m)/v$ be the ordered tree obtained by grafting the root of w_i to the i th leaf of v , where the nodes of v are greater than all nodes of w , but the relative orders within the w_i and v are maintained. When v is the ordered tree corresponding to 1432 and $w \xrightarrow{\gamma} (w_0, \dots, w_m)$ is the splitting (1.1), this grafting is

$$\begin{array}{c} \text{Diagram 8: A tree with root at bottom, left child has children 3, 2, 8, 11, right child has children 7, 5, 1, 10, 6, 9, 4.} \end{array} = \begin{array}{c} \text{Diagram 9: A tree with root at bottom, left child has children 3, 2, 8, 11, right child has children 7, 5, 1, 10, 6, 9, 4.} \end{array}$$

The notions of splitting and grafting also make sense for the unordered trees \mathcal{Y}_n and we use the same notation, $\bullet \xrightarrow{\gamma} \bullet$ and \bullet/\bullet . (Simply delete the labels in the constructions above.) These operations of splitting and grafting are compatible with the map $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$: if $w \xrightarrow{\gamma} (w_0, \dots, w_m)$ then $\tau(w) \xrightarrow{\gamma} (\tau(w_0), \dots, \tau(w_m))$ and all splittings in \mathcal{Y}_\bullet are induced in this way from splittings in \mathfrak{S}_\bullet . The same is true for grafting, $\tau((w_0, \dots, w_m)/v) = (\tau(w_0), \dots, \tau(w_m))/\tau(v)$.

1.2. $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$. For basics on Hopf algebras, see [19]. Let $\mathfrak{S}Sym := \bigoplus_{n \geq 0} \mathfrak{S}Sym_n$ be the graded \mathbb{Q} -vector space whose n^{th} graded piece has basis $\{F_w \mid w \in \mathfrak{S}_n\}$. Malvenuto and Reutenauer [16] defined a Hopf algebra structure on $\mathfrak{S}Sym$. For $w \in \mathfrak{S}_\bullet$, define the coproduct

$$\Delta F_w := \sum_{w \xrightarrow{\gamma} (w_0, w_1)} F_{w_0} \otimes F_{w_1},$$

where (w_0, w_1) is a pair of ordered trees. If $v \in \mathfrak{S}_m$, define the product

$$F_w \cdot F_v := \sum_{w \xrightarrow{\gamma} (w_0, \dots, w_m)} F_{(w_0, \dots, w_m)/v}.$$

The counit is the projection $\varepsilon: \mathfrak{S}Sym \rightarrow \mathfrak{S}Sym_0$ onto the 0th graded piece, which is spanned by the unit, $1 = F_\emptyset$, for this multiplication.

Proposition 1.1 ([16]). *With these definitions of coproduct, product, counit, and unit, $\mathfrak{S}Sym$ is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative.*

Let $\mathcal{YSym} := \bigoplus_{n \geq 0} \mathcal{YSym}_n$ be the graded \mathbb{Q} -vector space whose n^{th} graded piece has basis $\{F_t \mid t \in \mathcal{Y}_n\}$. Loday and Ronco [14] defined a Hopf algebra structure on \mathcal{YSym} . For $t \in \mathcal{Y}_\bullet$, define the coproduct

$$\Delta F_t := \sum_{t \xrightarrow{\mathcal{Y}} (t_0, t_1)} F_{t_0} \otimes F_{t_1},$$

and if $s \in \mathcal{Y}_m$, define the product

$$F_t \cdot F_s := \sum_{t \xrightarrow{\mathcal{Y}} (t_0, \dots, t_m)} F_{(t_0, \dots, t_m)/s}.$$

The counit is the projection $\varepsilon: \mathcal{YSym} \rightarrow \mathcal{YSym}_0$ onto the 0th graded piece, which is spanned by the unit, $1 = F_1$, for this multiplication. The map τ extends to a linear map $\tau: \mathfrak{S}Sym \rightarrow \mathcal{YSym}$, defined by $\tau(F_w) = F_{\tau(w)}$.

Proposition 1.2 ([14]). *With these definitions of coproduct, product, counit, and unit, \mathcal{YSym} is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative and the map τ a morphism of Hopf algebras.*

Some structures of the Hopf algebras $\mathfrak{S}Sym$ and \mathcal{YSym} , particularly their primitive elements and coradical filtrations are better understood with respect to a second basis. The Möbius function μ (or μ_P) of a poset P is defined for pairs (x, y) of elements of P with $\mu(x, y) = 0$ if $x \not\leq y$, $\mu(x, x) = 1$, and, if $x < y$, then

$$(1.2) \quad \mu(x, y) = - \sum_{x \leq z < y} \mu(x, z) \quad \text{so that} \quad 0 = \sum_{x \leq z \leq y} \mu(x, z).$$

For $w \in \mathfrak{S}_\bullet$ and $t \in \mathcal{Y}_\bullet$, set

$$(1.3) \quad M_w := \sum_{w \leq v} \mu(w, v) F_v \quad \text{and} \quad M_t := \sum_{t \leq s} \mu(t, s) F_s,$$

where the first sum is over $v \in \mathfrak{S}_\bullet$, the second sum over $s \in \mathcal{Y}_\bullet$, and $\mu(\cdot, \cdot)$ is the Möbius function in the weak and Tamari orders.

Proposition 1.3 ([1, 2]). *If $w \in \mathfrak{S}_\bullet$, then*

$$(1.4) \quad \tau(M_w) = \begin{cases} M_{\tau(w)}, & \text{if } w = \max(\tau(w)), \\ 0, & \text{otherwise} \end{cases}$$

and

$$(1.5) \quad \Delta M_w = \sum_{w=u \setminus v} M_u \otimes M_v.$$

If $t \in \mathcal{Y}_\bullet$, then

$$(1.6) \quad \Delta M_t = \sum_{t=r \setminus s} M_r \otimes M_s.$$

This implies that the set $\{M_w \mid w \in \mathfrak{S}_\bullet\}$ is indecomposable is a basis for the primitive elements of $\mathfrak{S}Sym$ (and the same for \mathcal{YSym}), thereby explicitly realizing the cofree-ness of $\mathfrak{S}Sym$ and \mathcal{YSym} .

1.3. Möbius functions and interval retracts. A pair $f: P \rightarrow Q$ and $g: Q \rightarrow P$ of poset maps is a *Galois connection* if f is left adjoint to g in that

$$\forall p \in P \text{ and } q \in Q, \quad f(p) \leq_Q q \iff p \leq_P g(q).$$

When this occurs, Rota [21, Theorem 1] related the Möbius functions of P and Q :

$$\forall p \in P \text{ and } q \in Q, \quad \sum_{f(y)=q} \mu_P(p, y) = \sum_{g(x)=q} \mu_Q(x, q).$$

Rota's formula was used in [2] to establish the coproduct formulas (1.4) and (1.6), as the maps $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$ and $\max: \mathcal{Y}_\bullet \rightarrow \mathfrak{S}_\bullet$ form a Galois connection [4, Section 9].

We do not have a Galois connection between \mathfrak{S}_\bullet and \mathcal{M}_\bullet , and so cannot use Rota's formula. Nevertheless, there is a useful relation between the Möbius functions of \mathfrak{S}_\bullet and \mathcal{M}_\bullet that we establish here in a general form. A surjective poset map $f: P \rightarrow Q$ from a finite lattice P is an *interval retract* if the fibers of f are intervals and if f admits an order-preserving section $g: Q \rightarrow P$ with $f \circ g = \text{id}$.

Theorem 1.4. *Let the poset map $f: P \rightarrow Q$ is an interval retract, then the Möbius functions μ_P and μ_Q of P and Q are related by the formula*

$$(1.7) \quad \mu_Q(x, y) = \sum_{\substack{f(a)=x \\ f(b)=y}} \mu_P(a, b) \quad (\forall x, y \in Q).$$

In Section 2, we define an interval retract $\beta: \mathfrak{S}_n \rightarrow \mathcal{M}_n$.

We evaluate each side of (1.7) using Hall's formula, which expresses the Möbius function in terms of *chains*. A linearly ordered subset $C: x_0 < \dots < x_r$ of a poset is a *chain* of *length* $\ell(C) = r$ from x_0 to x_r . Given a poset P , let $\mathfrak{C}(P)$ be the set of all chains in P . A poset P is an *interval* if it has a unique maximum element and a unique minimum element. If $P = [x, y]$ is an interval, let $\mathfrak{C}'(P)$ denote the chains in P beginning in x and ending in y . Hall's formula states that

$$\mu(x, y) = \sum_{C \in \mathfrak{C}'[x, y]} (-1)^{\ell(C)}.$$

Our proof rests on the following two lemmas.

Lemma 1.5. *If P is an interval, then $\sum_{C \in \mathfrak{C}(P)} (-1)^{\ell(C)} = 1$.*

Proof. Suppose that $P = [x, y]$ and append new minimum and maximum elements to P to get $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$. Then the definition of Möbius function (1.2) gives

$$\mu(\hat{0}, \hat{1}) = - \sum_{\hat{0} \leq z \leq y} \mu(\hat{0}, z),$$

which is zero by (1.2). By Hall's formula,

$$0 = \mu(\hat{0}, \hat{1}) = \sum_{C \in \mathfrak{C}'[\hat{0}, \hat{1}]} (-1)^{\ell(C)} = -1 + \sum_{C \in \mathfrak{C}(P)} (-1)^{\ell(C)+2},$$

where the term -1 comes from the chain $\hat{0} < \hat{1}$. This proves the lemma. \square

Call a partition $P = K_0 \sqcup \cdots \sqcup K_r$ of P into subposets K_i *monotone* if $x < y$ with $x \in K_i$ and $y \in K_j$ implies that $i \leq j$. Given $\emptyset \subsetneq I \subseteq [0, r]$, write $\mathfrak{U}_I(P)$ for the subset of chains C in $\mathfrak{U}(P)$ such that $C \cap K_i \neq \emptyset$ if and only if $i \in I$.

Lemma 1.6. *Let $P = K_0 \sqcup \cdots \sqcup K_r$ be a monotonic partition of a poset P . If $\bigcup_{i \in I} K_i$ is an interval for all $I \subseteq [0, r]$, then*

$$(1.8) \quad \sum_{C \in \mathfrak{U}_{[0, r]}(P)} (-1)^{\ell(C)} = (-1)^r.$$

Proof. We argue by induction on r . Lemma 1.5 is the case $r = 0$ (wherein $K_0 = P$), so we consider the case $r \geq 1$.

Form the poset $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ as in the proof of Lemma 1.5. Since P is an interval, we have $\sum_{C \in \mathfrak{U}'[\hat{0}, \hat{1}]} (-1)^{\ell(C)} = 0$. As $\mathfrak{U}'[\hat{0}, \hat{1}] = \bigsqcup_I \mathfrak{U}_I(P)$ we have,

$$0 = -1 + \sum_{\emptyset \subsetneq I \subsetneq [0, r]} \left(\sum_{C \in \mathfrak{U}_I(P)} (-1)^{\ell(C)} \right) + \sum_{C \in \mathfrak{U}_{[0, r]}(P)} (-1)^{\ell(C)},$$

where the term -1 counts the chain $\hat{0} < \hat{1}$. Applying induction, we have

$$0 = \sum_{k=0}^r \binom{r+1}{k} (-1)^{k-1} + \sum_{C \in \mathfrak{U}_{[0, r]}(P)} (-1)^{\ell(C)}.$$

Comparing this to the binomial expansion of $(1-1)^{r+1}$ completes the proof. \square

Proof of Theorem 1.4. Fix $x < y$ in Q . We use Hall's formula to rewrite the right-hand side of (1.7) as

$$(1.9) \quad \sum_{\substack{f(x)=a \\ f(y)=b}} \sum_{C \in \mathfrak{U}'[a, b]} (-1)^{\ell(C)}.$$

Fix a chain $D: q_0 < \cdots < q_r$ in $\mathfrak{U}'[x, y]$ and let $P|_D$ be the subposet of P consisting of elements that occur in some chain of P that maps to D under f . This is nonempty as f has section. Furthermore, the sets $K_i := f^{-1}(q_i) \cap P|_D$, for $i = 0, \dots, r$, form a monotonic partition of $P|_D$. We claim that $\bigcup_{i \in I} K_i$ is an interval for all $I \subseteq [0, r]$. If so, let us first rewrite (1.9) as a sum over chains D in Q ,

$$\sum_{D \in \mathfrak{U}'[x, y]} \sum_{C \in \mathfrak{U}_{[0, \ell(D)]}(P|_D)} (-1)^{\ell(C)}.$$

By Lemma 1.6, the inner sum becomes $\sum_D (-1)^{\ell(D)}$, which completes the proof.

To prove the claim, suppose that $I = \{i_0 < \cdots < i_s\}$. Each set K_i ($i \in I$) is an interval, as it is the intersection of two intervals in the lattice P . Thus K_{i_0} and K_{i_s} are intervals with minimum and maximum elements m and M , respectively. Any chain in $\bigcup_{i \in I} K_i$ can be extended to a chain beginning with m and ending at M , so $\bigcup_{i \in I} K_i$ is an interval. \square

2. THE MULTIPLIHEDRA \mathcal{M}_\bullet .

The map $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$ forgets the linear ordering of the node poset of an ordered tree, and it induces a morphism of Hopf algebras $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$. In fact, one may take the (ahistorical) view that the Hopf structure on $\mathcal{Y}Sym$ is induced from that on $\mathfrak{S}Sym$ via the map τ . Forgetting some, but not all, of the structure on a tree in \mathfrak{S}_\bullet factorizes the map τ . Here, we study combinatorial consequences of one such factorization, and later treat its algebraic consequences.

2.1. Bi-leveled trees. A *bi-leveled tree* $(t; \mathsf{T})$ is a planar binary tree $t \in \mathcal{Y}_n$ together with an (upper) order ideal T of its node poset, where T contains the leftmost node of t as a minimal element. Thus T contains all nodes along the path from the leftmost leaf to the root, and none above the leftmost node. Numbering the gaps between the leaves of t by $1, \dots, n$ from left to right, T becomes a subset of $\{1, \dots, n\}$.

Saneblidze and Umble [22] introduced bi-leveled trees to describe a cellular projection from the permutahedra to Stasheff's multiplihedra \mathcal{M}_\bullet , with the bi-leveled trees on n nodes indexing the vertices \mathcal{M}_n . Stasheff used a different type of tree for the vertices of \mathcal{M}_\bullet . These alternative trees lead to a different Hopf structure which we explore in a forthcoming paper [9]. We remark that $\mathcal{M}_0 = \{1\}$.

The partial order on \mathcal{M}_n is defined by $(s; \mathsf{S}) \leq (t; \mathsf{T})$ if $s \leq t$ in \mathcal{Y}_n and $\mathsf{S} \supseteq \mathsf{T}$. The Hasse diagrams of the posets \mathcal{M}_n are 1-skeleta for the multiplihedra. We represent a bi-leveled tree by drawing the underlying tree t and circling the nodes in T . The Hasse diagram of \mathcal{M}_4 appears in Figure 2.

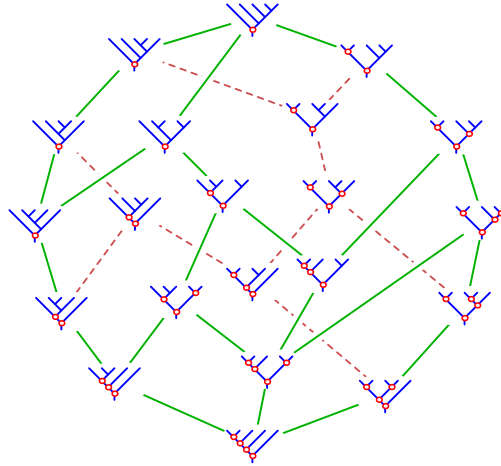


FIGURE 2. The 1-skeleton of the multiplihedron \mathcal{M}_4 .

2.2. Poset maps. Forgetting the order ideal in a bi-leveled tree, $(t; \mathsf{T}) \mapsto t$, is a poset map $\phi: \mathcal{M}_\bullet \rightarrow \mathcal{Y}_\bullet$. We define a map $\beta: \mathfrak{S}_\bullet \rightarrow \mathcal{M}_\bullet$ so that

$$\mathfrak{S}_\bullet \xrightarrow{\beta} \mathcal{M}_\bullet \xrightarrow{\phi} \mathcal{Y}_\bullet$$

factors the map $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$, and we define a right inverse (section) ι of β .

Let $w \in \mathfrak{S}_\bullet$ be an ordered tree. Define the set

$$(2.1) \quad \mathsf{T}(w) := \{i \mid w(i) \geq w(1)\}.$$

Observe that $(\tau(w); \mathsf{T}(w))$ is a bi-leveled tree. Indeed, as w is a linear extension of $\tau(w)$, $\mathsf{T}(w)$ is an upper order ideal which by definition (2.1) contains the leftmost node as a minimal element. Since covers in the weak order can only decrease the subset $\mathsf{T}(w)$ and τ is also a poset map, we see that β is a poset map.

Theorem 2.1. *The maps $\beta: \mathfrak{S}_\bullet \rightarrow \mathcal{M}_\bullet$ and $\phi: \mathcal{M}_\bullet \rightarrow \mathcal{Y}_\bullet$ are surjective poset maps with $\tau = \phi \circ \beta$.*

The fibers of the map β are intervals (indeed, products of intervals); see Figure 3. We prove this using an equivalent representation of a bi-leveled tree and

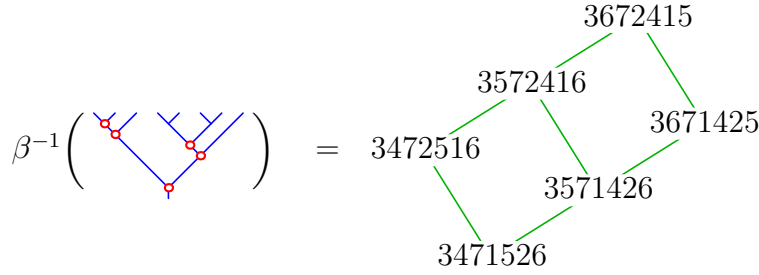


FIGURE 3. The preimages of β are intervals.

a description of the map β in that representation. If we prune a bi-leveled tree $b = (t; \mathsf{T})$ above the nodes in T (but not on the leftmost branch) we obtain a tree t'_0 (the order ideal) on r nodes and a planar forest $\mathbf{t} = (t_1, \dots, t_r)$ of r trees. If we prune t'_0 just below its leftmost node, we obtain the tree Y (from the pruning) and a tree t_0 , and t'_0 is obtained by grafting Y onto the leftmost leaf of t_0 . We may recover b from this tree t_0 on $r-1$ nodes and the planar forest $\mathbf{t} = (t_1, \dots, t_r)$, and so we also write $b = (t_0, \mathbf{t})$. We illustrate this correspondence in Figure 4.

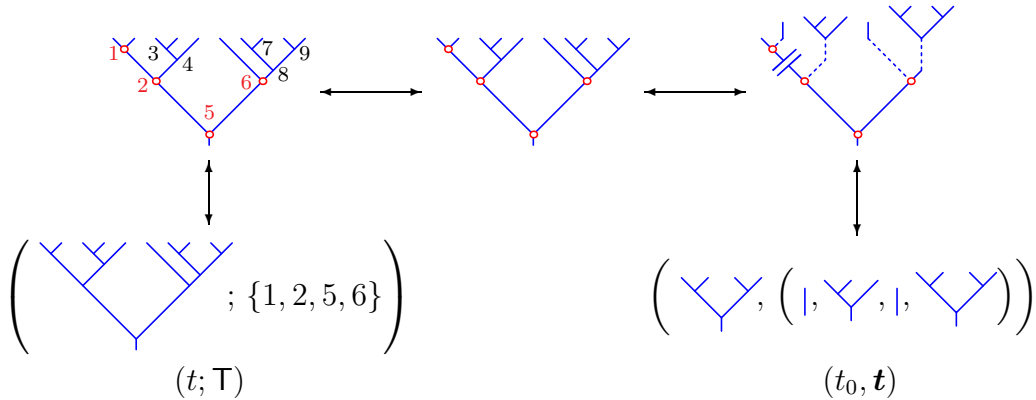


FIGURE 4. Two representations of bi-leveled trees.

We describe the map β in terms of this second representation of bi-leveled trees. Given a permutation w with $\beta(w) = (t; \mathsf{T})$ and $|\mathsf{T}| = r$, let $u_1 u_2 \dots u_r$ be the

restriction of w to the set T . We may write the values of w as $w = u_1 v^1 u_2 \cdots u_r v^r$, where v^i is the (possibly empty) subword of w between the numbers u_i and u_{i+1} and v^r is the word after u_r . Call this the *bi-leveled factorization* of w . For example,

$$654789132 \longmapsto (\underbrace{6}_{u_1}, \underbrace{54}_{v^1}, \underbrace{7}_{u_2}, \underbrace{\emptyset}_{v^2}, \underbrace{8}_{u_3}, \underbrace{\emptyset}_{v^3}, \underbrace{9}_{u_4}, \underbrace{132}_{v^4}).$$

Note that $\beta(w) = (\tau(\overline{u_2 \dots u_r}), (\tau(\overline{v^1}), \dots, \tau(\overline{v^r})))$.

Theorem 2.2. *For any $b \in \mathcal{M}_n$ the fiber $\beta^{-1}(b) \subseteq \mathfrak{S}_n$ is a product of intervals.*

Proof. Let $b = (t_0, (t_1, \dots, t_r)) = (t; \mathsf{T}) \in \mathcal{M}_n$ be a bi-leveled tree. A permutation $w \in \beta^{-1}(b) \in \mathfrak{S}_n$ has a bi-leveled factorization $w = u_1 v^1 u_2 \dots u_r v^r$ with

$$(2.2) \quad \begin{aligned} (i) \quad & w|_{\mathsf{T}} = u_1 u_2 \dots u_r, \quad u_1 = n+1-r, \quad \tau(\overline{u_2 \dots u_r}) = t_0, \quad \text{and} \\ (ii) \quad & \tau(\overline{v^i}) = t_i, \quad \text{for } i = 1, \dots, r. \end{aligned}$$

Since $u_1 < u_2, \dots, u_r$ are the values of w in the positions of T , and $u_1 = n+1-r$ exceeds all the letters in v^1, \dots, v^r , which are the values of w in the positions in the complement of T , these two parts of the bi-leveled factorization may be chosen independently to satisfy (2.2), which shows that $\beta^{-1}(b)$ is a product.

To see that the factors are intervals, and thus $\beta^{-1}(b)$ is an interval, we examine the conditions (i) and (ii) separately. Those $u_1 \dots u_r = w|_{\mathsf{T}}$ for w in the fiber $\beta^{-1}(b)$ are exactly the set of $n+1-r, u_2, \dots, u_r$ with $\{u_2, \dots, u_r\} = \{n+2-r, \dots, n\}$ and $\tau(\overline{u_2 \dots u_r}) = t_0$. This is a poset under the restriction of the weak order, and it is in natural bijection with the interval $\tau^{-1}(t_0) \subset \mathfrak{S}_{r-1}$. Its minimal element is $\min_0(b) = u_1 u_2 \dots u_r$, where $u_2 \dots u_r$ is the unique 231-avoiding word on $\{n+1-r, \dots, n\}$ satisfying (i), and its maximal element is $\max_0(b) = u_1 u_2 \dots u_r$, where now $u_2 \dots u_r$ is the unique 132-avoiding word on $\{n+1-r, \dots, n\}$ satisfying (i).

Now consider sequences of words v^1, \dots, v^r on distinct letters $\{1, \dots, n-r\}$ satisfying (ii). This is also a poset under the restriction of the weak order. It has a minimal element, which is the unique such sequence $\min(b)$ satisfying (ii) where the letters of v^i precede those of v^j whenever $i < j$, and where each v^i is 231-avoiding. Its maximal element is the unique sequence $\max(b)$ satisfying (ii) where the letters of v^i are greater than those of v^j when $i < j$ and v^i is 132-avoiding. \square

The fibers of β are intervals so that consistently choosing the minimum or maximum in a fiber gives two set-theoretic sections. These are not order-preserving as may be seen from Figure 5. We have $\text{min}(b) < \text{min}(c)$ but the maxima in their fibers

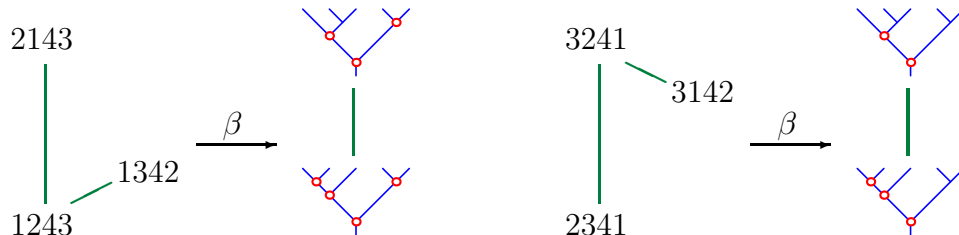


FIGURE 5. Fibers of β .

under β , 1342 and 2143, are incomparable. Similarly, $\begin{smallmatrix} \text{blue} & \text{red} \\ \swarrow & \searrow \\ \text{blue} & \text{red} \end{smallmatrix} < \begin{smallmatrix} \text{red} & \text{blue} \\ \swarrow & \searrow \\ \text{red} & \text{blue} \end{smallmatrix}$ but the minima in their fibers under β , 2341 and 3142, are incomparable. This shows that the map $\beta: \mathfrak{S}_\bullet \rightarrow \mathcal{M}_\bullet$ is not a lattice congruence (unlike the map $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$ [20]).

In the notation of the proof, given a bi-leveled tree $b = (t_0, (t_1, \dots, t_r))$, let $\iota(b)$ be the permutation $w \in \beta^{-1}(b)$ with bi-leveled factorization $w = u_1 v^1 u_2 \dots u_r v^r$ where $u_1 u_2 \dots u_r = \min_0(b)$ and $(v^1, \dots, v^r) = \max(b)$. This defines a map $\iota: \mathcal{M}_n \rightarrow \mathfrak{S}_n$ that is a section of the map β . For example,

$$\iota \left(\begin{array}{c} \text{blue} \quad \text{red} \quad \text{blue} \quad \text{red} \quad \text{blue} \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \text{red} \quad \text{blue} \quad \text{red} \quad \text{blue} \end{array} \right) = \begin{array}{c} 7 \ 8 \ 6 \ 11 \ 4 \ 5 \ 9 \ 10 \ 2 \ 3 \ 1 \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \text{blue} \quad \text{red} \quad \text{blue} \quad \text{red} \quad \text{blue} \end{array} = 786 \ 11 \ 459 \ 10 \ 231.$$

Remark 2.3. This map ι may be characterized in terms of pattern avoidance: the permutation $\iota(b)$ is the unique $w \in \beta^{-1}(b)$ avoiding the pinned patterns

$$\{\underline{2}031, \underline{0}231, \underline{3}021\},$$

where the underlined letter must be the first letter of a permutation. To see this, note that the first pattern forces the letters in v^i to be larger than those in v^{i+1} for $1 \leq i < r$, the second pattern forces $u_2 \dots u_r$ to be 231-avoiding, and the last pattern forces each v^i to be 132-avoiding.

Theorem 2.4. *The map ι is injective, right-inverse to β , and order-preserving. That is, $\beta: \mathfrak{S}_n \rightarrow \mathcal{M}_n$ is an interval retract.*

Since \mathfrak{S}_n is a lattice [10], the fibers of β are intervals, and ι is a section of β . That is, we need only verify that ι is order-preserving. We begin by describing the covers in \mathcal{M}_\bullet . Since β is a surjective poset map, every cover in \mathcal{M}_n is the image of some cover $w \lessdot w'$ in \mathfrak{S}_n .

Lemma 2.5. *If a cover $w \lessdot w' \in \mathfrak{S}_n$ does not collapse under β , i.e., $\beta(w) \neq \beta(w')$, then it yields one of three types of covers $\beta(w) \lessdot \beta(w')$ in \mathcal{M}_n .*

- (i) *In exactly one tree t_i in $\beta(w) = (t_0, (t_1, \dots, t_r))$, a node is moved from left to right across its parent to obtain $\beta(w')$. That is, $t_i \lessdot t'_i$.*
- (ii) *If $\beta(w) = (t; \mathbb{T})$, the leftmost node of t is moved across its parent, which has no other child in the order ideal \mathbb{T} , and is deleted from \mathbb{T} to obtain $\beta(w')$.*
- (iii) *If $\mathbb{T}(w) = \{1 = T_1 < \dots < T_r\}$, then $\tau(w') = \tau(w)$ and $\mathbb{T}(w') = \mathbb{T}(w) \setminus \{T_j\}$ for some $j > 2$.*

Proof. Put $w' = (k, k+1)w$, with $k, k+1$ appearing in order in w . Let $(t; \mathbb{T})$ and $(t_0, (t_1, \dots, t_r))$ be the two representations of $\beta(w)$. Write $\mathbb{T} = \{T_1 < \dots < T_r\}$ (with $T_1 = 1$) and $w|_{\mathbb{T}} = u_1 u_2 \dots u_r$. If $w \lessdot w'$ and $\beta(w) \lessdot \beta(w')$, then k appears within w in one of three ways: (i) $u_1 \neq k$, (ii) $u_1 = k$ and $u_2 = k+1$, or (iii) $u_1 = k$ and $u_j = k+1$ for some $j > 2$. These yield the corresponding descriptions in the statement of the lemma. (Note that in type (i), $\mathbb{T}(w') = \mathbb{T}$, so if we set $\beta(w') = (t'_0, (t'_1, \dots, t'_r))$, then $t_i = t'_i$, except for one index i , where $t_i \lessdot t'_i$.) \square

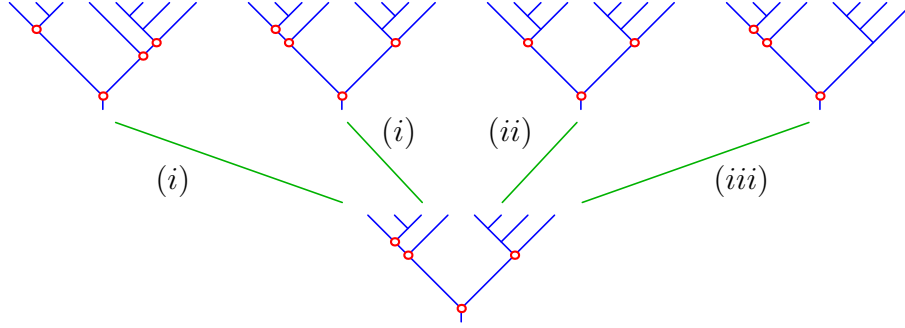
FIGURE 6. Some covers in \mathcal{M}_7 .

Figure 6 illustrates these three types of covers, labeled by their type.

For $\mathsf{T} \subset \{1, \dots, n\}$ with $1 \in \mathsf{T}$, let $\mathfrak{S}_n(\mathsf{T}) := \{w \in \mathfrak{S}_n \mid \mathsf{T}(w) = \mathsf{T}\}$. Let $\mathcal{M}_n(\mathsf{T})$ be those bi-leveled trees whose order ideal consists of the nodes in T . Note that $\beta(\mathfrak{S}_n(\mathsf{T})) = \mathcal{M}_n(\mathsf{T})$ and $\beta^{-1}(\mathcal{M}_n(\mathsf{T})) = \mathfrak{S}_n(\mathsf{T})$.

Lemma 2.6. *The map $\iota: \mathcal{M}_n(\mathsf{T}) \rightarrow \mathfrak{S}_n(\mathsf{T})$ is a map of posets.*

Proof. Let $\mathsf{T} = \{1 = T_1 < \dots < T_r\}$. Setting $T_{r+1} = n+1$, define $a_i := T_{i+1} - T_i - 1$ for $i = 1, \dots, r$. Then $b \mapsto (t_0, (t_1, \dots, t_r))$ gives an isomorphism of posets,

$$\mathcal{M}_n(\mathsf{T}) \xrightarrow{\sim} \mathcal{Y}_{r-1} \times \mathcal{Y}_{a_1} \times \dots \times \mathcal{Y}_{a_r}.$$

As the maps $\min, \max: \mathcal{Y}_a \rightarrow \mathfrak{S}_a$ are order-preserving, the proof of Theorem 2.2 gives the desired result. \square

Proof of Theorem 2.4. Let $b \triangleleft c$ be a cover in \mathcal{M}_n . We will show that $\iota(b) \leq \iota(c)$ in \mathfrak{S}_n . Suppose that $b = (t; \mathsf{T})$, with $\mathsf{T} = \{1 = T_1 < \dots < T_r\}$. Let $\iota(b)$ have bi-leveled factorization $\iota(b) = u_1 v^1 u_2 v^2 \dots u_r v^r$, and set $k := n+1 - |\mathsf{T}|$.

The result is immediate if the cover $b \triangleleft c$ is of type (i), for then $b, c \in \mathcal{M}_n(\mathsf{T})$ and $\iota: \mathcal{M}_n(\mathsf{T}) \rightarrow \mathfrak{S}_n$ is order-preserving, as observed in Lemma 2.6.

Now suppose that $b \triangleleft c$ is a cover of type (ii). Set $w := \iota(b)$. We claim that $w \triangleleft (k, k+1)w$ and $\iota(c) = (k, k+1)w$. Now, $u_1 = k$ labels the leftmost node of b , so the first claim is immediate. Note that u_2 labels the parent of the node labeled b . This parent has no other child in T , so we must have $u_2 < u_3$. As $u_2 u_3 \dots u_r$ is 231-avoiding and contains $k+1$, we must have $u_2 = k+1$. This shows that

$$\iota(c) = (k, k+1)w = u_2 (v^1 u_1 v^2) u_3 \dots u_r v^r.$$

Indeed, u_2 is minimal among u_2, \dots, u_r and $u_3 \dots u_r$ is 231-avoiding, thus $\min_0(c) = u_2 \dots u_r$. The bi-leveled factorization of $(k, k+1)w$ gives $(v^1 u_1 v^2, v^3, \dots, v^r)$, which we claim is $\max(c)$. As u_1 is the largest letter in the sequence, we need only check that $v^1 u_1 v^2$ is 132-avoiding. But this is true for v^1 and v^2 and there can be no 132-pattern involving u_1 as the letters in v^1 are all greater than those in v^2 .

Finally, suppose that $b \triangleleft c$ is of type (iii). Then $c = (t; \mathsf{T} \setminus \{T_j\})$ for some $j > 2$. We will find a permutation $w' \in \beta^{-1}(b)$ satisfying $(k, k+1)w' \in \beta^{-1}(c)$ and

$$(2.3) \quad \iota(b) \leq w' \triangleleft (k, k+1)w' \leq \iota(c).$$

Let $w' \in \beta^{-1}(b)$ be the minimal permutation having bi-leveled factorization

$$w' = u'_1 v^1 u'_2 \dots u'_r v^r, \quad \text{with} \quad u'_j = k+1.$$

Here $(v^1, \dots, v^r) = \underline{\max}(b)$ is the same sequence as in $\iota(b)$. The structure of $\beta^{-1}(b)$ implies that $\iota(b) \leq w'$. We also have

$$w' \leq (k, k+1)w' \quad \text{and} \quad \beta((k, k+1)w') = c.$$

While $\iota(c)$ and $(k, k+1)w'$ are not necessarily equal, we do have that

$$(k, k+1)w'|_{\tau \setminus \{T_j\}} = u'_j u'_2 \dots u'_{j-1} u'_{j+1} \dots u'_r$$

and $u'_2 \dots u'_{j-1} u'_{j+1} \dots u'_r$ is 231-avoiding. That is, $(k, k+1)w'|_{\tau \setminus \{T_j\}} = \iota(c)|_{\tau \setminus \{T_j\}}$. Otherwise, w' would not be minimal. The bi-leveled factorization of $(k, k+1)w'$ is

$$u'_j v^1 u'_2 \dots u'_{j-1} (v^{j-1} u'_1 v^j) u'_{j+1} \dots u'_r v^r,$$

and we necessarily have $(v^1, \dots, v^{j-1} u'_1 v^j, \dots, v^r) \leq \underline{\max}(c)$, which implies that $(k, k+1)w' \leq \iota(c)$. We thus have the chain (2.3) in \mathfrak{S}_n , completing the proof. \square

If $b \leq c$ is the cover of type (iii) in Figure 6, the chain (2.3) from $\iota(b)$ to $\iota(c)$ is

$$4357126 \leq 4367125 \leq 5367124 \leq 5467123.$$

2.3. Tree enumeration. Let

$$\mathbf{S}(q) := \sum_{n \geq 0} n! q^n = 1 + q + 2q^2 + 6q^3 + 24q^4 + 120q^5 + \dots$$

be the enumerating series of permutations, and define $\mathbf{M}(q)$ and $\mathbf{Y}(q)$ similarly

$$(2.4) \quad \begin{aligned} \mathbf{M}(q) &:= \sum_{n \geq 0} A_n q^n = 1 + q + 2q^2 + 6q^3 + 21q^4 + 80q^5 + \dots, \\ \mathbf{Y}(q) &:= \sum_{n \geq 0} C_n q^n = 1 + q + 2q^2 + 5q^3 + 14q^4 + 42q^5 + \dots, \end{aligned}$$

where $A_n := |\mathcal{M}_n|$ and $C_n := |\mathcal{Y}_n|$ are the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$, whose enumerating series satisfies

$$\mathbf{Y}(q) = \frac{1 - \sqrt{1 - 4q}}{2q} = \frac{2}{1 + \sqrt{1 - 4q}}.$$

Bi-leveled trees are Catalan-like [8, Theorem 3.1]: for $n \geq 1$, $A_n = C_{n-1} + \sum_{i=1}^{n-1} A_i A_{n-i}$. See also [24, A121988]. Their enumerating series satisfies

$$\mathbf{M}(q) = 1 + q\mathbf{Y}(q) \cdot \mathbf{Y}(q\mathbf{Y}(q)).$$

We will also be interested in $\mathbf{M}_+(q) := \sum_{n \geq 0} A_n q^n = q\mathbf{Y}(q) \cdot \mathbf{Y}(q\mathbf{Y}(q))$.

Theorem 2.7. *The only nontrivial quotients of the enumerating series $\mathbf{S}(q)$, $\mathbf{M}(q)$, $\mathbf{M}_+(q)$, and $\mathbf{Y}(q)$ whose expansions have nonnegative coefficients are*

$$\mathbf{S}(q)/\mathbf{M}(q), \quad \mathbf{S}(q)/\mathbf{Y}(q), \quad \mathbf{M}_+(q)/\mathbf{Y}(q), \quad \text{and} \quad \mathbf{M}(q)/\mathbf{Y}(q).$$

Proof. We prove the positivity of the quotient $\mathbf{S}(q)/\mathbf{M}(q)$ in Section 4.2. The positivity of $\mathbf{S}(q)/\mathbf{Y}(q)$ was established after [2, Theorem 7.2], which shows that $\mathfrak{S}Sym$ is a smash product over $\mathcal{Y}Sym$.

For the positivity of $\mathbf{M}_+(q)/\mathbf{Y}(q)$, we use [3, Proposition 3], which computes $\mathbf{Y}(q\mathbf{Y}(q)) = \sum_{n \geq 0} B_n q^{n-1}$, where

$$(2.5) \quad B_1 := C_0 \quad \text{and} \quad B_n := \sum_{k=0}^{n-1} \frac{k}{n-1} \binom{2n-k-3}{n-k-1} C_k \quad \text{for } n > 1.$$

In particular, $B_n \geq 0$ for all $n \geq 0$. Returning to the quotient, we have

$$\frac{\mathbf{M}_+(q)}{\mathbf{Y}(q)} = \frac{q\mathbf{Y}(q) \cdot \mathbf{Y}(q\mathbf{Y}(q))}{\mathbf{Y}(q)} = q\mathbf{Y}(q\mathbf{Y}(q)),$$

so $\mathbf{M}_+(q)/\mathbf{Y}(q) = \sum_{n \geq 0} B_n q^n$ has nonnegative coefficients.

For $\mathbf{M}(q)/\mathbf{Y}(q)$, use the identity $1/\mathbf{Y}(q) = 1 - q\mathbf{Y}(q)$ to obtain

$$\frac{\mathbf{M}(q)}{\mathbf{Y}(q)} = \mathbf{M}_+(q) + 1 - q\mathbf{Y}(q) = 1 + \sum_{n \geq 0} (B_n - C_{n-1}) q^n.$$

Positivity is immediate as $B_n - C_{n-1} \geq 0$ for $n > 0$.

We leave the proof that the remaining quotients have negative coefficients to the reader's computer. \square

Remark 2.8. Up to an index shift, the quotient $\mathbf{M}_+(q)/\mathbf{Y}(q)$ corresponds to the sequence [24, A127632] beginning with $(1, 1, 3, 11, 44, 185, 804)$. We give a new combinatorial interpretation of this sequence in Corollary 4.3.

3. THE ALGEBRA \mathcal{MSym}

Let $\mathcal{MSym} := \bigoplus_{n \geq 0} \mathcal{MSym}_n$ denote the graded \mathbb{Q} -vector space whose n^{th} graded piece has the basis $\{F_b \mid b \in \mathcal{M}_n\}$. The maps $\beta: \mathfrak{S}_\bullet \rightarrow \mathcal{M}_\bullet$ and $\phi: \mathcal{M}_\bullet \rightarrow \mathcal{Y}_\bullet$ of graded sets induce surjective maps of graded vector spaces

$$(3.1) \quad \mathfrak{S}Sym \xrightarrow{\beta} \mathcal{MSym} \xrightarrow{\phi} \mathcal{Y}Sym \quad F_w \mapsto F_{\beta(w)} \mapsto F_{\phi(\beta(w))},$$

which factor the Hopf algebra map $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$, as $\phi(\beta(w)) = \tau(w)$. We will show how the maps β and τ induce on \mathcal{MSym} the structures of an algebra, of a $\mathfrak{S}Sym$ -module, and of a $\mathcal{Y}Sym$ -comodule so that the composition (3.1) factors the map τ as maps of algebras, of $\mathfrak{S}Sym$ -modules, and of $\mathcal{Y}Sym$ -comodules.

3.1. Algebra structure on \mathcal{MSym} . For $b, c \in \mathcal{M}_\bullet$ define

$$(3.2) \quad F_b \cdot F_c = \beta(F_w \cdot F_v),$$

where w, v are permutations in \mathfrak{S}_\bullet with $b = \beta(w)$ and $c = \beta(v)$.

Theorem 3.1. *The operation $F_b \cdot F_c$ defined by (3.2) is independent of choices of w, v with $\beta(w) = b$ and $\beta(v) = c$ and it endows \mathcal{MSym} with the structure of a graded connected algebra such that the map $\beta: \mathfrak{S}Sym \rightarrow \mathcal{MSym}$ is a surjective map of graded connected algebras.*

If the expression $\beta(F_w \cdot F_v)$ is independent of choice of $w \in \beta^{-1}(b)$ and $v \in \beta^{-1}(c)$, then the map β is automatically multiplicative. Associative and unital properties for \mathcal{MSym} are then inherited from those for $\mathfrak{S}\mathcal{Sym}$, and the theorem follows. To prove independence (in Lemma 3.2), we formulate a description of (3.2) in terms of splittings and graftings of bi-leveled trees.

Let $s \xrightarrow{\gamma} (s_0, \dots, s_m)$ be a splitting on the underlying tree of a bi-leveled tree $b = (s; \mathbf{S}) \in \mathcal{M}_n$. Then the nodes of s are distributed among the nodes of the partially ordered forest (s_0, \dots, s_m) so that the order ideal \mathbf{S} gives a sequence of order ideals in the trees s_i . Write $b \xrightarrow{\gamma} (b_0, \dots, b_m)$ for the corresponding splitting of the bi-leveled tree b , viewing b_i as $(s_i; \mathbf{S}|_{s_i})$. (Note that only b_0 is guaranteed to be a bi-leveled tree.) Given $c = (t; \mathbf{T}) \in \mathcal{M}_m$ and a splitting $b \xrightarrow{\gamma} (b_0, \dots, b_m)$ of $b \in \mathcal{M}_n$, form a bi-leveled tree $(b_0, \dots, b_m)/c$ whose underlying tree is $(s_0, \dots, s_m)/t$ and whose order ideal is either

$$(3.3) \quad \begin{aligned} (i) & \quad \mathbf{T}, \text{ if } b_0 \in \mathcal{M}_0, \text{ or} \\ (ii) & \quad \mathbf{S} \cup \{\text{the nodes of } t\}, \text{ if } b_0 \notin \mathcal{M}_0. \end{aligned}$$

Lemma 3.2. *The product (3.2) is independent of choices of w, v with $\beta(w) = b$ and $\beta(v) = c$. For $b \in \mathcal{M}_n$ and $c \in \mathcal{M}_m$, we have*

$$F_b \cdot F_c = \sum_{b \xrightarrow{\gamma} (b_0, \dots, b_m)} F_{(b_0, \dots, b_m)/c}.$$

Proof. Fix any $w \in \beta^{-1}(b)$ and $v \in \beta^{-1}(c)$. The bi-leveled tree $\beta((w_0, \dots, w_m)/v)$ associated to a splitting $w \xrightarrow{\gamma} (w_0, \dots, w_m)$ has underlying tree $(s_0, \dots, s_m)/t$, where $s \xrightarrow{\gamma} (s_0, \dots, s_m)$ is the induced splitting on the underlying tree $s = \tau(w) = \phi(b)$. Each node of $(w_0, \dots, w_m)/v$ comes from a node of either w or v , with the labels of nodes from w all smaller than the labels of nodes from v . Consequently, the leftmost node of $(w_0, \dots, w_m)/v$ comes from either

- (i) v , and then $\mathbf{T}((w_0, \dots, w_m)/v) = \mathbf{T}(v) = \mathbf{T}(c)$, or
- (ii) w , and then $\mathbf{T}((w_0, \dots, w_m)/v) = \mathbf{T}(w) = \mathbf{T}(b) \cup \{\text{the nodes of } v\}$.

The first case is when $w_0 \in \mathfrak{S}_0$ and the second case is when $w_0 \notin \mathfrak{S}_0$. \square

Here is the product $F_{\swarrow} \cdot F_{\swarrow}$, together with the corresponding splittings of \swarrow ,

$$F_{\swarrow} \cdot F_{\swarrow} = F_{\swarrow \swarrow} + F_{\swarrow \nwarrow} + F_{\nwarrow \swarrow} + F_{\nwarrow \nwarrow} + F_{\swarrow \swarrow} + F_{\swarrow \nwarrow}.$$

3.2. $\mathfrak{S}\mathcal{Sym}$ module structure on \mathcal{MSym} . Since β is a surjective algebra map, \mathcal{MSym} becomes a $\mathfrak{S}\mathcal{Sym}$ -bimodule with the action

$$F_w \cdot F_b \cdot F_v = F_{\beta(w)} \cdot F_b \cdot F_{\beta(v)}.$$

The map τ likewise induces on $\mathcal{Y}\mathcal{Sym}$ the structure of a $\mathfrak{S}\mathcal{Sym}$ -bimodule, and the maps β , ϕ , and τ are maps of $\mathfrak{S}\mathcal{Sym}$ -bimodules.

Curiously, we may use the map $\iota: \mathcal{M}_\bullet \rightarrow \mathfrak{S}_\bullet$ to define the structure of a right $\mathfrak{S}Sym$ -comodule on \mathcal{MSym} ,

$$F_b \longmapsto \sum_{\iota(b) \xrightarrow{\gamma} (w_0, w_1)} F_{\beta(w_0)} \otimes F_{w_1}.$$

This induces a right comodule structure, because if $\iota(b) \xrightarrow{\gamma} (w_0, w_1)$, then $w_0 = \iota(\beta(w_0))$, which may be checked using the characterization of ι in terms of pattern avoidance, as explained in Remark 2.3.

While \mathcal{MSym} is both a right $\mathfrak{S}Sym$ -module and right $\mathfrak{S}Sym$ -comodule, it is not an $\mathfrak{S}Sym$ -Hopf module. For if it were a Hopf module, then the fundamental theorem of Hopf modules (see Remark 4.4) would imply that the series $\mathbf{M}(q)/\mathbf{S}(q)$ has positive coefficients, which contradicts Theorem 2.7.

3.3. $\mathcal{Y}Sym$ -comodule structure on \mathcal{MSym} . For $b \in \mathcal{M}_\bullet$, define the linear map $\rho: \mathcal{MSym} \rightarrow \mathcal{MSym} \otimes \mathcal{Y}Sym$ by

$$(3.4) \quad \rho(F_b) = \sum_{b \xrightarrow{\gamma} (b_0, b_1)} F_{b_0} \otimes F_{\phi(b_1)}.$$

By $\phi(b_1)$, we mean the tree underlying b_1 .

Example 3.3. In the fundamental bases of \mathcal{MSym} and $\mathcal{Y}Sym$, we have

$$\rho(F_{\text{V}}) = F_{\text{V}} \otimes 1 + F_{\text{V}} \otimes F_{\text{V}} + F_{\text{V}} \otimes F_{\text{V}} + F_{\text{V}} \otimes F_{\text{V}} + 1 \otimes F_{\text{V}}.$$

Theorem 3.4. *Under ρ , \mathcal{MSym} is a right $\mathcal{Y}Sym$ -comodule.*

Proof. This is counital as (b, \mathbf{l}) is a splitting of b . Coassociativity is also clear as both $(\rho \otimes 1)\rho$ and $(1 \otimes \Delta)\rho$ applied to F_b for $b \in \mathcal{M}_\bullet$ are sums of terms $F_{b_0} \otimes F_{\phi(b_1)} \otimes F_{\phi(b_2)}$ over all splittings $b \xrightarrow{\gamma} (b_0, b_1, b_2)$. \square

Careful bookkeeping of the terms in $\rho(F_b \cdot F_c)$ show that it equals $\rho(F_b) \cdot \rho(F_c)$ for all $b, c \in \mathcal{M}_\bullet$ and thus \mathcal{MSym} is a $\mathcal{Y}Sym$ -comodule algebra. Hence, ϕ is a map of $\mathcal{Y}Sym$ -comodule algebras, and in fact β is also a map of $\mathcal{Y}Sym$ -comodule algebras. We leave this to the reader, and will not pursue it further.

Since $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$ is a map of Hopf algebras, $\mathfrak{S}Sym$ is naturally a right $\mathcal{Y}Sym$ -comodule where the comodule map is the composition

$$\mathfrak{S}Sym \xrightarrow{\Delta} \mathfrak{S}Sym \otimes \mathfrak{S}Sym \xrightarrow{1 \otimes \tau} \mathfrak{S}Sym \otimes \mathcal{Y}Sym.$$

With these definitions, the following lemma is immediate.

Lemma 3.5. *The maps τ and ϕ are maps of right $\mathcal{Y}Sym$ -comodules.*

In particular, we have the equality of maps $\mathfrak{S}Sym \rightarrow \mathcal{MSym} \otimes \mathcal{Y}Sym$,

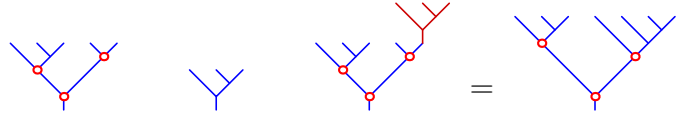
$$(3.5) \quad \rho \circ \beta = (\beta \otimes \tau) \circ \Delta.$$

3.4. Coaction in the monomial basis. The coalgebra structures of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ were elucidated by considering a second basis related to the fundamental basis via Möbius inversion. For $b \in \mathcal{M}_n$, define

$$(3.6) \quad M_b := \sum_{b \leq c} \mu(b, c) F_c,$$

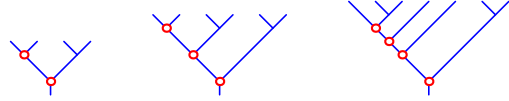
where $\mu(\cdot, \cdot)$ is the Möbius function on the poset \mathcal{M}_n .

Given $b \in \mathcal{M}_m$ and $s \in \mathcal{Y}_q$, write $b \setminus s$ for the bi-leveled tree with $p + q$ nodes whose underlying tree is formed by grafting the root of s onto the rightmost leaf of b , but whose order ideal is that of b . Here is an example of b , s , and $b \setminus s$,



Observe that we cannot have $b = \mathbf{I}$ in this construction.

The maximum bi-leveled tree with a given underlying tree t is $\beta(\max(t))$, which has order ideal \mathbf{T} consisting only of the nodes of t along its leftmost branch. Here are three such trees of the form $\beta(\max(t))$,



Theorem 3.6. *Given $b = (t; \mathbf{T}) \in \mathcal{M}_\bullet$, we have*

$$\rho(M_b) = \begin{cases} \sum_{b=c \setminus s} M_c \otimes M_s & \text{if } b \neq \beta(\max(t)) \\ \sum_{b=c \setminus s} M_c \otimes M_s + 1 \otimes M_t & \text{if } b = \beta(\max(t)) \end{cases}.$$

For example,

$$\begin{aligned} \rho(M_{\text{tree 1}}) &= M_{\text{tree 1}} \otimes 1 \\ \rho(M_{\text{tree 2}}) &= M_{\text{tree 2}} \otimes 1 + M_{\text{tree 2}} \otimes M_{\text{tree 3}} \\ \rho(M_{\text{tree 3}}) &= M_{\text{tree 3}} \otimes 1 + M_{\text{tree 3}} \otimes M_{\text{tree 4}} + M_{\text{tree 4}} \otimes M_{\text{tree 5}} + 1 \otimes M_{\text{tree 6}}. \end{aligned}$$

Our proof of Theorem 3.6 uses Proposition 1.3 and the following results.

Lemma 3.7. *For any bi-leveled tree $b \in \mathcal{M}_\bullet$, we have*

$$\beta\left(\sum_{\beta(w)=b} M_w\right) = M_b.$$

Proof. Expand the left hand side in terms of the fundamental bases to get

$$\beta\left(\sum_{\beta(w)=b} \sum_{w \leq v} \mu_{\mathfrak{S}}(w, v) F_v\right) = \sum_{\beta(w)=b} \sum_{w \leq v} \mu_{\mathfrak{S}}(w, v) F_{\beta(v)}.$$

As β is surjective, we may change the index of summation to $b \leq c$ in \mathcal{M}_\bullet to obtain

$$\sum_{b \leq c} \left(\sum_{\substack{\beta(w)=b \\ \beta(v)=c}} \mu_{\mathfrak{S}}(w, v) \right) F_c.$$

By Theorems 1.4 and 2.4, the inner sum is $\mu_{\mathcal{M}}(b, c)$, so this sum is M_b . \square

Recall that $w = u \setminus v$ only if $\tau(w) = \tau(u) \setminus \tau(v)$ and the values of w in the nodes of u exceed the values in the nodes of v . We always have the trivial decomposition $w = (\emptyset, w)$. Suppose that $w = u \setminus v$ with $u \neq \emptyset$ a nontrivial decomposition. If $\beta(w) = b = (t; \mathsf{T})$, then T is a subset of the nodes of u so that $\beta(u) = (\tau(u); \mathsf{S})$ and $b = \beta(u) \setminus \tau(v)$. Moreover, for every decomposition $b = c \setminus s$ and every u, v with $\beta(u) = c$ and $\tau(v) = s$, we have $b = \beta(u \setminus v)$. Thus, for $b \in \mathcal{M}_\bullet$, we have

$$(3.7) \quad \bigsqcup_{\beta(w)=b} \bigsqcup_{\substack{w=u \setminus v \\ u \neq \emptyset}} (u, v) = \bigsqcup_{b=c \setminus t} \bigsqcup_{\beta(u)=c} \bigsqcup_{\tau(v)=t} (u, v).$$

Proof of Theorem 3.6. Let $b = (t; \mathsf{T})$ with $t \neq \mathbf{1}$. Using Lemma 3.7, we have

$$\rho(M_b) = \rho\beta \left(\sum_{\beta(w)=b} M_w \right) = \sum_{\beta(w)=b} \rho\beta M_w.$$

By (3.5), (3.7), and (1.5), this equals

$$\begin{aligned} & \sum_{\beta(w)=b} \sum_{\substack{w=u \setminus v \\ u \neq \emptyset}} \beta(M_u) \otimes \tau(M_v) + \sum_{\beta(w)=b} \beta(M_\emptyset) \otimes \tau(M_w) \\ &= \sum_{b=c \setminus s} \left(\sum_{\beta(u)=c} \beta(M_u) \right) \otimes \left(\sum_{\tau(v)=s} \tau(M_t) \right) + \sum_{\beta(w)=b} 1 \otimes \tau(M_w). \end{aligned}$$

By Lemma 3.7 and (1.4), the first sum becomes $\sum_{b=c \setminus s} M_c \otimes M_s$ and the second sum vanishes unless $b = \beta(\max(t))$. This completes the proof. \square

4. HOPF VARIATIONS

4.1. The $\mathcal{Y}Sym$ -Hopf module \mathcal{MSym}_+ . Let $\mathcal{M}_+ := (\mathcal{M}_n)_{n \geq 1}$ be the bi-leveled trees with at least one internal node and define \mathcal{MSym}_+ to be the positively graded part of \mathcal{MSym} , which has bases indexed by \mathcal{M}_+ . A *restricted splitting* of $b \in \mathcal{M}_+$ is a splitting $b \xrightarrow{\mathsf{Y}_+} (b_0, \dots, b_m)$ with $b_0 \in \mathcal{M}_+$, i.e., $b_0 \neq \mathbf{1}$. Given $b \xrightarrow{\mathsf{Y}_+} (b_0, \dots, b_m)$ and $t \in \mathcal{Y}_m$, form the bi-leveled tree $(b_0, \dots, b_m)/t$ by grafting the ordered forest (b_0, \dots, b_m) onto the leaves of t , with order ideal consisting of the nodes of t together with the nodes of the forest coming from the order ideal of b , as in (3.3)(ii).

We define an action and coaction of $\mathcal{Y}Sym$ on \mathcal{MSym}_+ that are similar to the product and coaction on \mathcal{MSym} . They come from a second collection of polytope

maps $\mathcal{M}_n \rightarrow \mathcal{Y}_{n-1}$ arising from viewing the vertices of \mathcal{M}_n as *painted trees* on $n-1$ nodes (see [5, 8]). For $b \in \mathcal{M}_+$ and $t \in \mathcal{Y}_m$, set

$$(4.1) \quad \begin{aligned} F_b \cdot F_t &= \sum_{b \xrightarrow{\Upsilon_+} (b_0, \dots, b_m)} F_{(b_0, \dots, b_m)/t}, \\ \rho_+(F_b) &= \sum_{b \xrightarrow{\Upsilon_+} (b_0, b_1)} F_{b_0} \otimes F_{\phi(b_1)}. \end{aligned}$$

For example, in the fundamental bases of \mathcal{MSym}_+ and \mathcal{YSym} , we have

$$\begin{aligned} F_{\text{V}} \cdot F_{\text{V}} &= F_{\text{V}} + F_{\text{V}} + F_{\text{V}}, \\ \rho_+(F_{\text{V}}) &= F_{\text{V}} \otimes 1 + F_{\text{V}} \otimes F_{\text{V}} + F_{\text{V}} \otimes F_{\text{V}} + F_{\text{V}} \otimes F_{\text{V}}. \end{aligned}$$

Theorem 4.1. *The operations in (4.1) define a \mathcal{YSym} -Hopf module structure on \mathcal{MSym}_+ .*

Proof. The unital and counital properties are immediate. We check only that the action is associative, the coaction is coassociative, and the two structures commute with each other.

Associativity. Fix $b = (t; \mathsf{T}) \in \mathcal{M}_+$, $r \in \mathcal{Y}_m$, and $s \in \mathcal{Y}_n$. A term in the expression $(F_b \cdot F_r) \cdot F_s$ corresponds to a restricted splitting and grafting $b \xrightarrow{\Upsilon_+} (b_0, \dots, b_m) \rightsquigarrow (b_0, \dots, b_m)/r = c$, followed by another $c \xrightarrow{\Upsilon_+} (c_0, \dots, c_n) \rightsquigarrow (c_0, \dots, c_n)/t$. The order ideal for this term equals $\mathsf{T} \cup \{\text{the nodes of } r \text{ and } s\}$. Note that restricted splittings of c are in bijection with pairs of splittings

$$\left(b \xrightarrow{\Upsilon_+} (b_0, \dots, b_{m+n}), r \xrightarrow{\Upsilon} (r_0, \dots, r_n) \right).$$

Terms of $F_b \cdot (F_r \cdot F_s)$ also correspond to these pairs of splittings. The order ideal for this term is again $\mathsf{T} \cup \{\text{the nodes of } r \text{ and } s\}$. That is, $(F_b \cdot F_r) \cdot F_s$ and $F_b \cdot (F_r \cdot F_s)$ agree term by term.

Coassociativity. Fix $b = (t; \mathsf{T}) \in \mathcal{M}_+$. Terms $F_c \otimes F_r \otimes F_s$ in $(\rho_+ \otimes \mathbb{1})\rho_+(F_b)$ and $(\mathbb{1} \otimes \Delta)\rho_+(F_b)$ both correspond to restricted splittings $b \xrightarrow{\Upsilon_+} (c, c_1, c_2)$, where $\phi(c_1) = r$ and $\phi(c_2) = s$. In either case, the order ideal on c is $\mathsf{T}|_c$.

Commuting structures. Fix $b = (s; \mathsf{S}) \in \mathcal{M}_+$ and $t \in \mathcal{Y}_m$. A term $F_{c_0} \otimes F_{\phi(c_1)}$ in $\rho_+(F_b \cdot F_t)$ corresponds to a choice of a restricted splitting and grafting $b \xrightarrow{\Upsilon_+} (b_0, \dots, b_m) \rightsquigarrow (b_0, \dots, b_r)/t = c$, followed by a restricted splitting $c \xrightarrow{\Upsilon_+} (c_0, c_1)$. The order ideal on c_0 equals the nodes of c_0 inherited from S , together with the nodes of c_0 inherited from t . The restricted splittings of c are in bijection with pairs of splittings $(b \xrightarrow{\Upsilon_+} (b_0, \dots, b_{m+1}), t \xrightarrow{\Upsilon} (t_0, t_1))$. If $t_0 \in \mathcal{Y}_n$, then the pair of graftings $c_0 = (b_0, \dots, b_n)/t_0$ and $c_1 = (b_{n+1}, \dots, b_m)/t_1$ are precisely the terms appearing in $\rho_+(F_b) \cdot \Delta(F_t)$. \square

The similarity of (4.1) to the coaction (3.4) of \mathcal{YSym} on \mathcal{MSym} gives the following result, whose proof we leave to the reader.

Corollary 4.2. *For $b \in \mathcal{M}_+$, we have*

$$\rho_+(M_b) = \sum_{b=c \setminus s} M_c \otimes M_s.$$

This elucidates the structure of \mathcal{MSym}_+ . Let $\mathcal{B} \subset \mathcal{M}_+$ be the indecomposable bi-leveled trees—those with only trivial decompositions, $b = b \setminus \mathbf{1}$. Then $(t; \mathbf{T}) \in \mathcal{B}$ if and only if \mathbf{T} contains the rightmost node of t . Every tree c in \mathcal{M}_+ has a unique decomposition $c = b \setminus s$ where $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$. Indeed, pruning c immediately above the rightmost node in its order ideal gives a decomposition $c = b \setminus s$ where $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$. This induces a bijection of graded sets,

$$\mathcal{M}_+ \longleftrightarrow \mathcal{B} \times \mathcal{Y}_\bullet.$$

Moreover, if $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$, then Corollary 4.2 and (1.6) together imply that

$$(4.2) \quad \rho_+(M_{b \setminus s}) = \sum_{s=r \setminus t} M_{b \setminus r} \otimes M_t.$$

Note that $\mathbb{Q}\mathcal{B} \otimes \mathcal{YSym}$ is a graded right \mathcal{YSym} -comodule with structure map,

$$b \otimes M_s \longmapsto b \otimes (\Delta M_s),$$

for $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\bullet$. Comparing this with (4.2), we deduce the following algebraic and combinatorial facts.

Corollary 4.3. *The map $\mathbb{Q}\mathcal{B} \otimes \mathcal{YSym} \rightarrow \mathcal{MSym}_+$ defined by $b \otimes M_s \mapsto M_{b \setminus s}$ is an isomorphism of graded right \mathcal{YSym} comodules.*

The quotient of enumerating series $\mathbf{M}(q)_+/\mathbf{Y}(q)$ is equal to the enumerating series of the graded set \mathcal{B} .

In particular, if $\mathcal{B}_n := \mathcal{B} \cap \mathcal{M}_n$, then $|\mathcal{B}_n| = B_n$ by (2.5).

Remark 4.4. The *coinvariants* in a right comodule M over a coalgebra C are $M^{\text{co}} := \{m \in M \mid \rho(m) = m \otimes 1\}$. We identify the vector space $\mathbb{Q}\mathcal{B}$ with $\mathcal{MSym}_+^{\text{co}}$ via $b \mapsto M_b$. The isomorphism $\mathbb{Q}\mathcal{B} \otimes \mathcal{YSym} \rightarrow \mathcal{MSym}_+$ is a special case of the Fundamental Theorem of Hopf Modules [19, Theorem 1.9.4]: If M is a Hopf module over a Hopf algebra H , then $M \simeq M^{\text{co}} \otimes H$ as Hopf modules.

4.2. Hopf module structure on \mathcal{MSym} . We use Theorem 3.6 to identify the \mathcal{YSym} -coinvariants in \mathcal{MSym} . Let \mathcal{B}' be those indecomposable bi-leveled trees which are not of the form $\beta(\max(t))$, for some $t \in \mathcal{Y}_+$, together with $\{\mathbf{1}\}$.

Corollary 4.5. *The \mathcal{YSym} -coinvariants of \mathcal{MSym} have a basis $\{M_b \mid b \in \mathcal{B}'\}$.*

For $n > 0$, the difference $\mathcal{B}_n \setminus \mathcal{B}'_n$ consists of indecomposable bi-leveled trees with n nodes of the form $\beta(\max(t))$. If $\beta(\max(t)) \in \mathcal{B}_n$, then $t = s \vee \mathbf{1}$, for some $s \in \mathcal{Y}_{n-1}$, and so $|\mathcal{B}'_n| = B_n - C_{n-1}$, which we saw in the proof of Theorem 2.7.

For $t \in \mathcal{Y}_\bullet$, set $\mathbf{1} \setminus t := \beta(\max(t))$, and if $\mathbf{1} \neq b \in \mathcal{B}'$, set $b \setminus t := b \setminus t$. Every bi-leveled tree uniquely decomposes as $b \setminus t$ with $b \in \mathcal{B}'$ and $t \in \mathcal{Y}_\bullet$. By Theorem 3.6, $M_b \otimes M_t \mapsto M_{b \setminus t}$ induces an isomorphism of right \mathcal{YSym} -comodules,

$$(4.3) \quad \mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym} \longrightarrow \mathcal{MSym},$$

where the structure map on $\mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym}$ is $M_b \otimes M_t \mapsto M_b \otimes \Delta(M_t)$. Treating $\mathcal{MSym}^{\text{co}}$ as a trivial \mathcal{YSym} -module, $M_b \cdot M_t = \varepsilon(M_t)M_b$, $\mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym}$ becomes a right \mathcal{YSym} -module. As explained in [19, Example 1.9.3], this makes $\mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym}$ into a \mathcal{YSym} -Hopf module.

We express this structure on \mathcal{MSym} . Let $b \setminus t \in \mathcal{M}_\bullet$ and $s \in \mathcal{Y}_\bullet$, then

$$(4.4) \quad M_{b \setminus t} \cdot M_s = \sum_{r \in t \cdot s} M_{b \setminus r} \quad \text{and} \quad \rho(M_{b \setminus t}) = \sum_{t=r \setminus s} M_{b \setminus r} \otimes M_s,$$

where $t \cdot s$ is the set of trees r indexing the product $M_t \cdot M_s$ in \mathcal{YSym} . The coaction is as before, but the product is new. It is not positive in the fundamental basis,

$$F_{\setminus} \cdot F_{\setminus} = F_{\setminus \setminus} - F_{\setminus \setminus} + F_{\setminus \setminus} + 2F_{\setminus \setminus}.$$

We complete the proof of Theorem 2.7.

Corollary 4.6. *The power series $\mathbf{S}(q)/\mathbf{M}(q)$ has nonnegative coefficients.*

Proof. Observe that

$$\mathbf{S}(q)/\mathbf{M}(q) = \left(\mathbf{S}(q)/\mathbf{Y}(q) \right) / \left(\mathbf{M}(q)/\mathbf{Y}(q) \right).$$

Since both $\mathfrak{S}\text{Sym}$ and \mathcal{MSym} are right \mathcal{YSym} -Hopf modules, the two quotients of enumerating series on the right are generating series for their coinvariants, by the Fundamental Theorem of Hopf modules. Thus

$$\mathbf{S}(q)/\mathbf{M}(q) = \mathbf{S}^{\text{co}}(q)/\mathbf{M}^{\text{co}}(q),$$

where $\mathbf{S}^{\text{co}}(q)$ and $\mathbf{M}^{\text{co}}(q)$ are the enumerating series for $\mathfrak{S}\text{Sym}^{\text{co}}$ and $\mathcal{MSym}^{\text{co}}$. To show that $\mathbf{S}^{\text{co}}(q)/\mathbf{M}^{\text{co}}(q)$ is nonnegative, we index bases for these spaces by graded sets \mathcal{S} and \mathcal{B}' , then establish a bijection $\mathcal{B}' \times \mathcal{S}' \rightarrow \mathcal{S}$ for some graded subset $\mathcal{S}' \subset \mathcal{S}$.

The set \mathcal{B}' was identified in Corollary 4.5. The coinvariants $\mathfrak{S}\text{Sym}^{\text{co}}$ were given in [2, Theorem 7.2] as a *left Hopf kernel*. The basis was identified as follows. Recall that permutations $u \in \mathfrak{S}_\bullet$ may be written uniquely in terms of indecomposables,

$$(4.5) \quad u = u_1 \setminus \cdots \setminus u_r$$

(taking $r = 0$ for $u = \emptyset$). Let $\mathcal{S} \subset \mathfrak{S}_\bullet$ be those permutations u whose rightmost indecomposable component has a 132-pattern, and thus $u \neq \max(t)$ for any $t \in \mathcal{Y}_+$. (Note that $u = \emptyset \in \mathcal{S}$.) Then $\{M_u \mid u \in \mathcal{S}\}$ is a basis for $\mathfrak{S}\text{Sym}^{\text{co}}$.

Fix a section $g: \mathcal{M}_\bullet \rightarrow \mathfrak{S}_\bullet$ of the map $\beta: \mathfrak{S}_\bullet \rightarrow \mathcal{M}_\bullet$ and define a subset $\mathcal{S}' \subset \mathcal{S}$ as follows. Given the decomposition $u = u_1 \setminus \cdots \setminus u_r$ in (4.5) with $r \geq 0$, consider the length $\ell \geq 0$ of the maximum initial sequence $u_1 \setminus \cdots \setminus u_\ell$ of indecomposables belonging to $g(\mathcal{B}')$. Put $u \in \mathcal{S}'$ if ℓ is even. Define the map of graded sets

$$\kappa: \mathcal{B}' \times \mathcal{S}' \longrightarrow \mathcal{S} \quad \text{by} \quad (b, v) \longmapsto g(b) \setminus v.$$

The image of κ lies in \mathcal{S} as the last component of a nontrivial $g(b) \setminus v$ is either $g(b)$ or the last component of v , neither of which can be $\max(t)$ for $t \in \mathcal{Y}_+$.

We claim that κ is bijective. If $u \in \mathcal{S}'$, then $u = \kappa(\mathbb{I}, u)$. If $u \in \mathcal{S} \setminus \mathcal{S}'$, then u has an odd number of initial components from $g(\mathcal{B}')$. Letting its first factor be $g(b)$,

we see that $u = g(b) \setminus u' = \kappa(b, u')$ with $u' \in \mathcal{S}'$. This surjective map is injective as the expressions $\kappa(l, u')$ and $\kappa(b, u')$ with $b \in \mathcal{B}'_+$ and $u' \in \mathcal{S}'$ are unique.

This isomorphism of graded sets identifies the enumerating series of the graded set \mathcal{S}' as the quotient $\mathbf{S}^{\text{co}}(q)/\mathbf{M}^{\text{co}}(q)$, which completes the proof. \square

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